IE5203 Decision Analysis
Solutions to Chapter 6 Exercises

P6.1

(*a*) $u(w) = w - \beta w^2$

$u'(w) = 1 - 2\beta w$
$u''(w) = -2\beta$

Risk tolerance $\rho(w) = \dfrac{-u'(w)}{u''(w)} = \dfrac{1-2\beta w}{2\beta}$

Degree of absolute risk aversion $r(w) = \dfrac{2\beta}{1-2\beta w}$

Note that for $u(w)$ to be increasing and concave (i.e., risk averse), it is sufficient for the coefficient β to be strictly positive.

(*b*) $u(w) = ln\ w$

$u'(w) = 1/w$
$u''(w) = -1/w^2$

Risk tolerance $\rho(w) = \dfrac{-u'(w)}{u''(w)} = w$

Degree of risk aversion = $r(w) = \dfrac{1}{w}$

(*c*) $u(w) = sgn(\beta)\, w^{\beta}$

$u'(w) = sgn(\beta)\, \beta\, w^{\beta-1}$
$u''(w) = sgn(\beta)\, \beta\, (\beta - 1)\, w^{\beta-2}$

Risk tolerance $\rho(w) = \dfrac{-u'(w)}{u''(w)} = \dfrac{w}{1-\beta}$

Degree of absolute risk aversion = $r(w) = \dfrac{1-\beta}{w}$

P6.2 Given that John has the utility function $u(x) = 1 - 3^{-x/50}$ over the range of x = -\$50 to \$5,000.

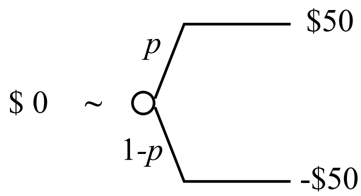
The utility function may be rewritten as $u(x) = 1 - 3^{-x/50} = 1 - e^{-x \ln 3/50}$

(*a*) The utility function is increasing and concave. Hence John is risk averse.

(*b*) John's risk tolerance = \$ 50 / *ln* 3.

Hence John's degree of risk aversion = 1 / risk tolerance = *ln* 3 / 50 = 0.0219722 $\$^{-1}$

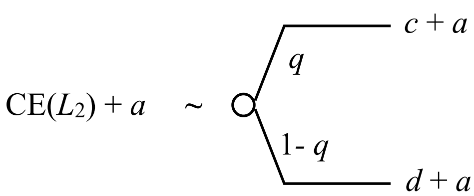
(*c*) We want to find p such that the certainty equivalent of the deal is zero.



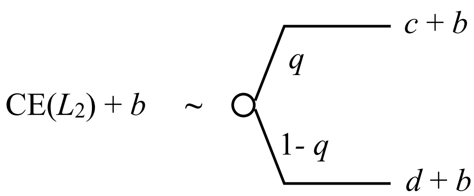
$$
\begin{aligned}
u(0) &= p\, u(50) + (1-p)\, u(-50) \\
1 - 3^0 &= p\,(1 - 3^{-1}) + (1-p)\,(1 - 3^1) \\
0 &= p\,(2/3) + (1-p)(-2) \\
p &= 3/4
\end{aligned}
$$

P6.3 Consider L_2:

By the delta property, if we add the amount a to all its outcomes of L_2, we get

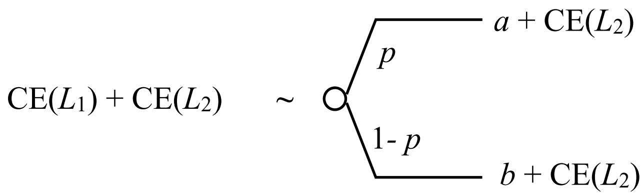


Similarly, by the Delta property, if we add the amount b to all the outcomes of L_2, we get



Consider L_1:

By the delta property, if we add the amount CE(L_2) to all its outcomes, we get



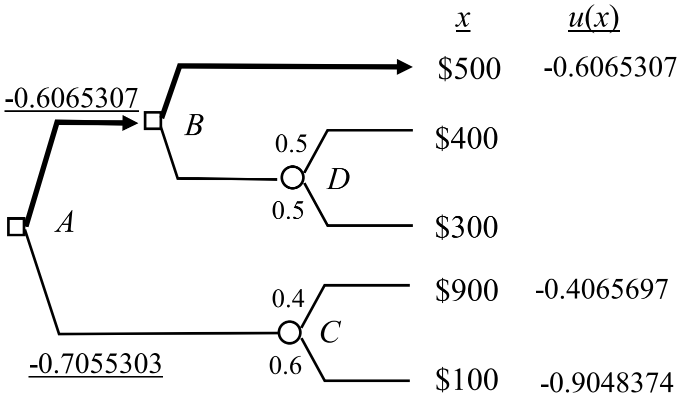
Finally, by the Substitution Rule, the above deal is equivalent to L_3.

Hence CE(L_3) = CE(L_1) + CE(L_2).

P6.4

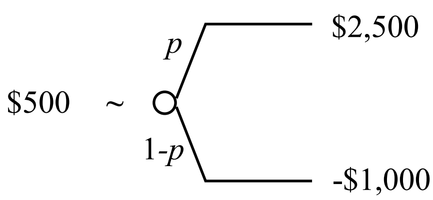
Since George has a constant risk tolerance of \$1,000, it follows that he has constant absolute risk aversion or the delta property. Hence his utility function is exponential in form.

We let $u(x) = -\exp(-x/1{,}000)$.



Maximum Expected Utility at A = -0.6065307
Certainty Equivalent at $A = u^{-1}(-0.60653)$ = \$500

The required preference probability is p, such that



Hence $\quad u(500) = p\, u(2{,}500) + (1-p)\, u(-1{,}000)$

$$\Rightarrow \quad -e^{\frac{-500}{1,000}} = (p)(-e^{\frac{-2,500}{1,000}}) + (1-p)(-e^{\frac{1,000}{1,000}})$$

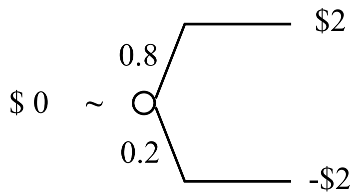
$$\Rightarrow \quad p = 0.80106$$

Note that we would have got the same answer if we had used the utility function $u(x) = 1 - exp(-x/1000)$, or if we had assumed $u(x) = a - b\, exp(-x/1000)$ and fitted the constants a and b to the boundary conditions $u(-\$1{,}000) = 0$ and $u(\$2{,}500) = 1$.

P6.5

(*a*) Susan satisfies delta property $\Rightarrow$ utility function is of the form $u(x) = a - b\, e^{-x/\rho}$ where ρ is the risk tolerance.

Given



$u(0) \;=\; 0.8\, u(2) \quad + 0.2\, u(-2)$

$a - b \;=\; 0.8\,(a - b\, e^{-2/\rho}) + 0.2\,(a - b\, e^{2/\rho})$

$1 = \; 0.8\, e^{-2/\rho} \quad + 0.2\, e^{2/\rho}$

Let $\quad x = e^{2/\rho}$

$$x^2 - 5x + 4 = 0 \;\Rightarrow\; (x-1)(x-4) = 0 \;\Rightarrow\; x = 1 \quad \text{or} \quad x = 4.$$

Hence $= e^{2/\rho} = 4 \;\Rightarrow\; \rho = \dfrac{1}{\ln 2} = \$1.44$

Susan risk tolerance = $1.44

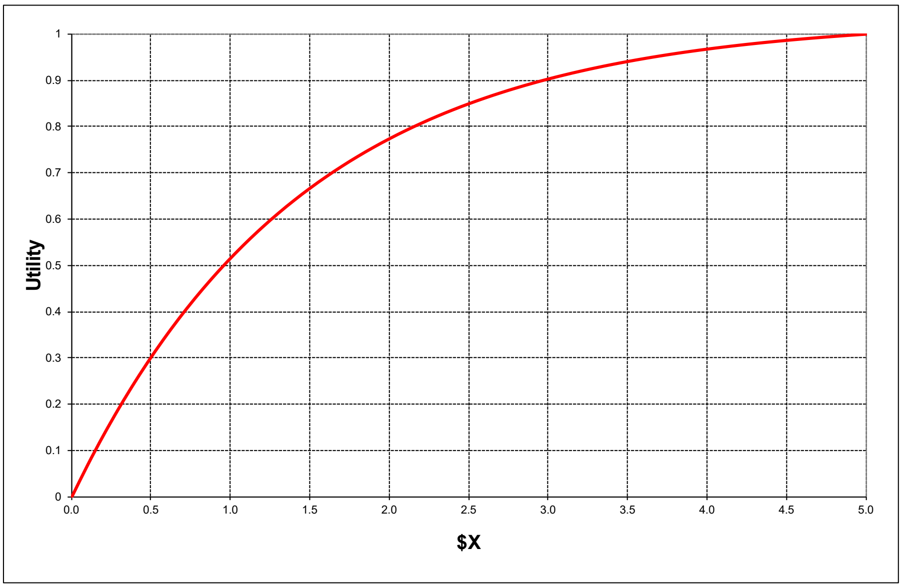
(*b*) Susan's risk attitude is risk averse since her risk tolerance is positive.

(*c*) Susan's utility function such that $u(L = 0) = 0$ and $u(H = 5) = 1$ is

$$u(x) = \frac{1 - e^{-(x-L)/\rho}}{1 - e^{-(H-L)/\rho}}$$

$$= 1.0322581\;(1 - e^{-x\ln 2})$$

$$= 1.0322581\;(1 - 2^{-x})$$

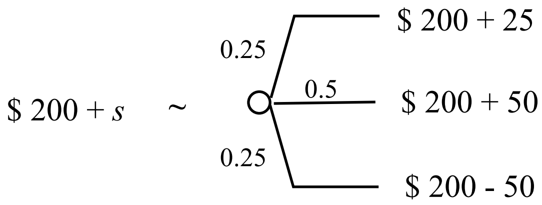


P6.6

Current wealth = \$200.

Wealth Utility function $\qquad u(w) = \dfrac{w^2}{2000}, \; w \geq 0.$

(*a*) Let s = personal indifferent selling price.



$u(200 + s) = 0.25\, u(200+25) + 0.5\, u(200 + 50\,) + 0.25\, u(200 - 50)$

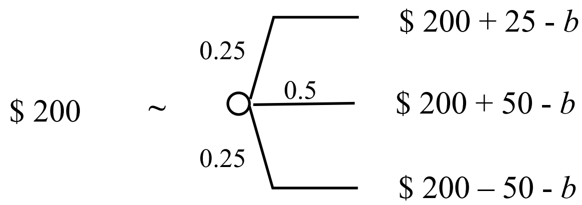
$$\frac{(200+s)^2}{2000} = 0.25\left(\frac{225^2}{2000}\right) + 0.5\left(\frac{250^2}{2000}\right) + 0.25\left(\frac{150^2}{2000}\right)$$

$$(200+s)^2 = 0.25\,(225)^2 + 0.5\,(250)^2 + 0.25\,(150)^2$$

$$s = \$\,22.5562$$

Hence Susan's PISP = **<u>\$22.56</u>**

(*b*) Let b = personal indifferent buying price.



$u(200) = 0.25\, u(200+25 - b) + 0.5\, u(200 + 50 - b) + 0.25\, u(200 - 50 - b)$

$$\frac{200^2}{2000} = 0.25\left(\frac{(225-b)^2}{2000}\right) + 0.5\left(\frac{(250-b)^2}{2000}\right) + 0.25\left(\frac{(150-b)^2}{2000}\right)$$

$$4(200)^2 = (225-b)^2 + 2(250-b)^2 + (150-b)^2$$

$$4b^2 - 1750b + 38125 = 0$$

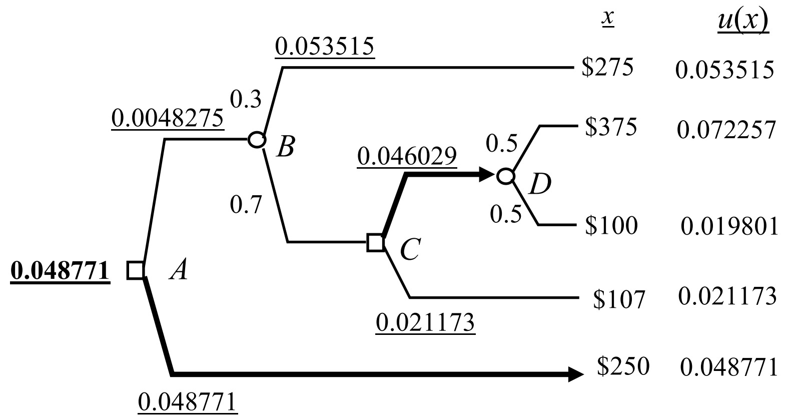
Solving: b = \$22.99425 (okay) or \$414.5057 (rejected)

Hence Susan's PIBP = **<u>\$22.99</u>**

P6.7 Note that the risk neutral case for this problem was covered in Chapter 4 Exercises.

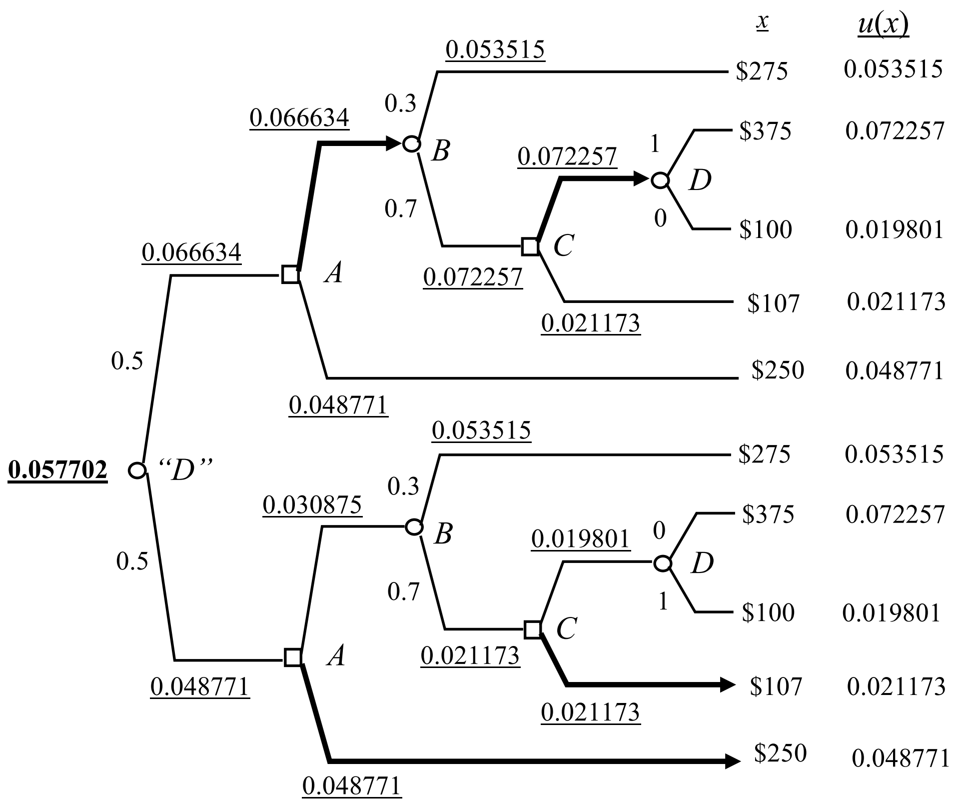
(*a*) Rex with $u(x) = 1 - e^{\frac{-x}{5000}}$, satisfies the delta property.

i. Base Decision model:



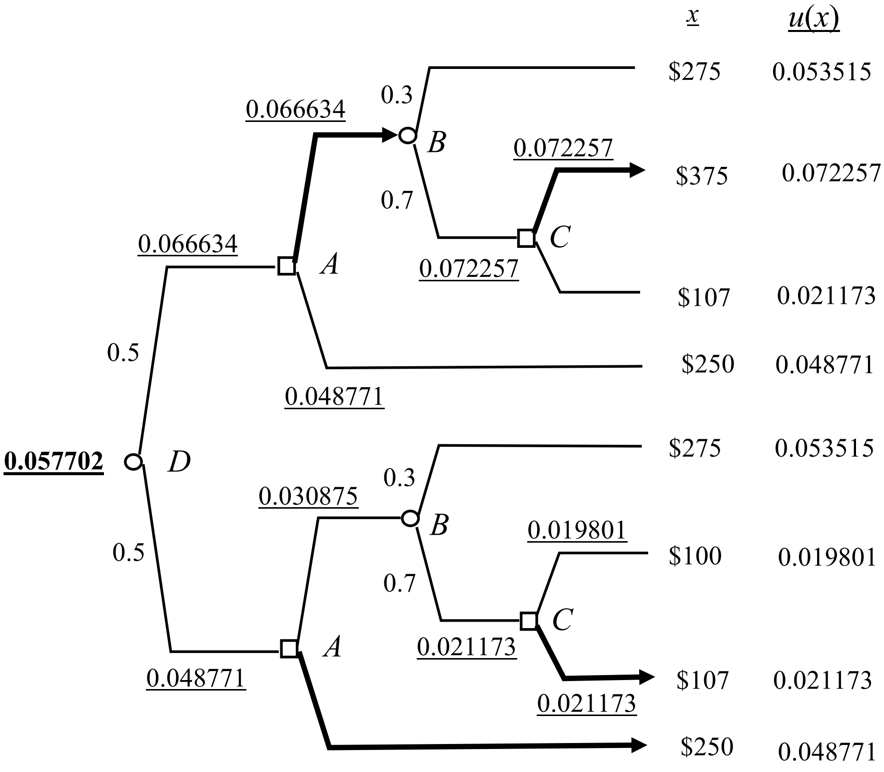
- Expected Utility at A = 0.048771
- Certainty Equivalent = u^{-1} (0.048771) = \$250.00

ii. Decision model with clairvoyance on *D* before making decision *A*.

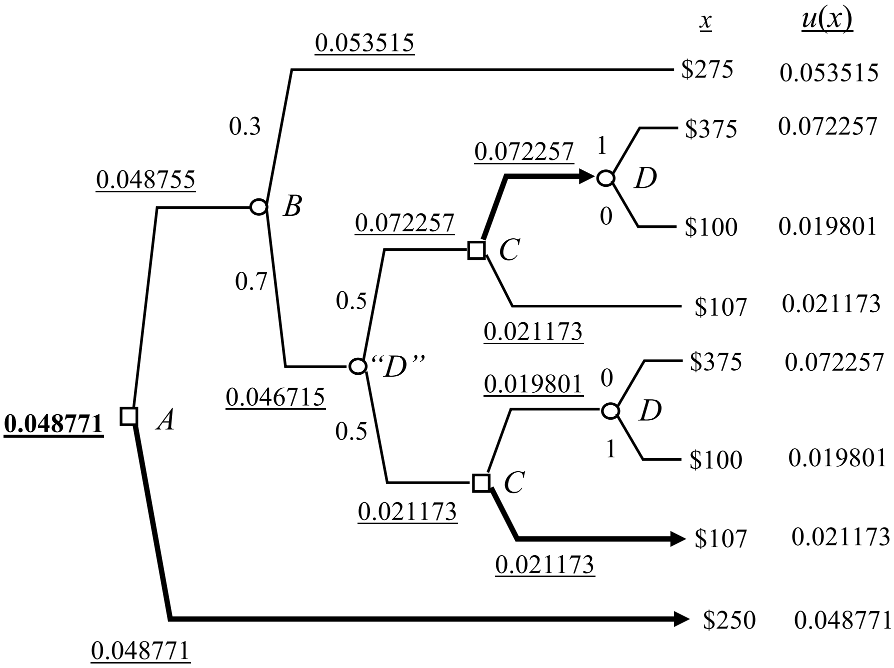


- Expected Utility of free clairvoyance on *D* before *A* = 0.057702
- Certainty Equivalent = u^{-1} (0.057702) = \$297.17
- Therefore, Value of clairvoyance on *D* before *A* = \$ 297.17 – 250.00 = \$47.17

Simplified decision model without 0-1 probability chance node.



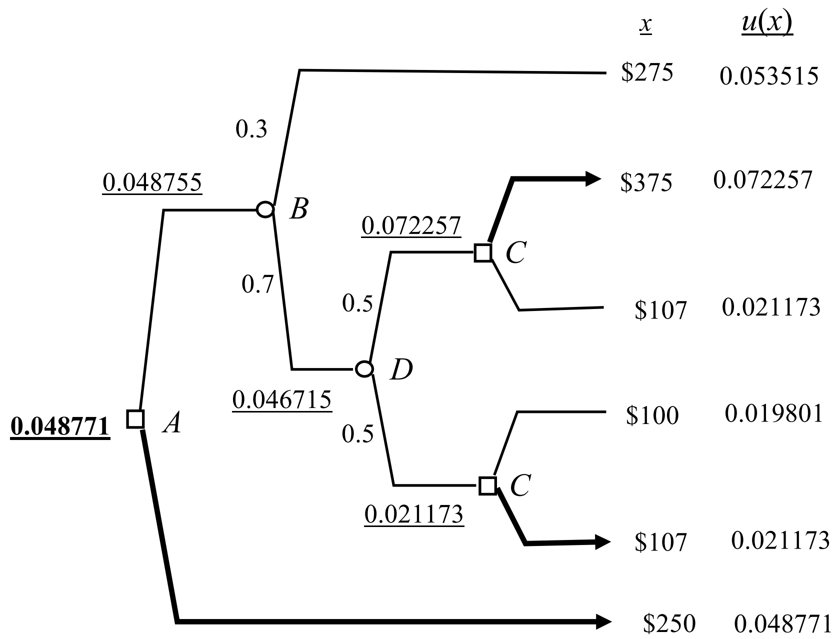
iii. Decision model with clairvoyance on *D* before decision *C* but after decision *A*.



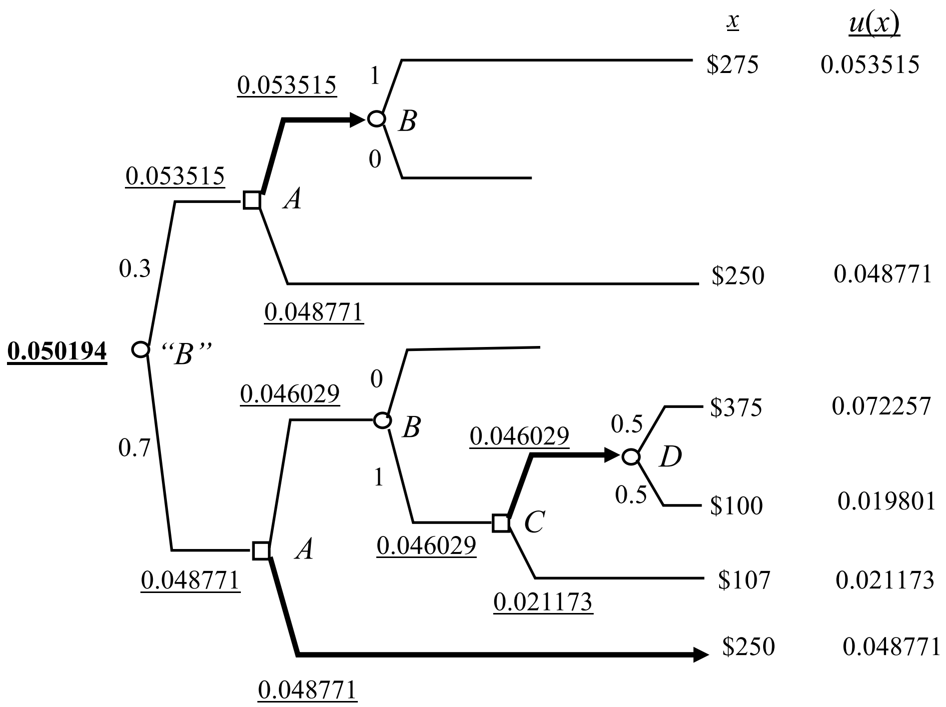
- Expected Utility of free clairvoyance on *D* before decision *C* but after *A* = 0.048771
- Certainty Equivalent = u^{-1} (0.048771) = \$250.00
- Therefore, value of clairvoyance on *D* before *C* but after decision *A* = \$250 - \$250 = \$ 0

The value of clairvoyance is zero because Rex will make exactly the same decisions here as he did without any information, i.e., the optimal decision policy is the same, with and without information. Hence the value of information is ZERO.

Simplified decision model without 0-1 probability chance node.

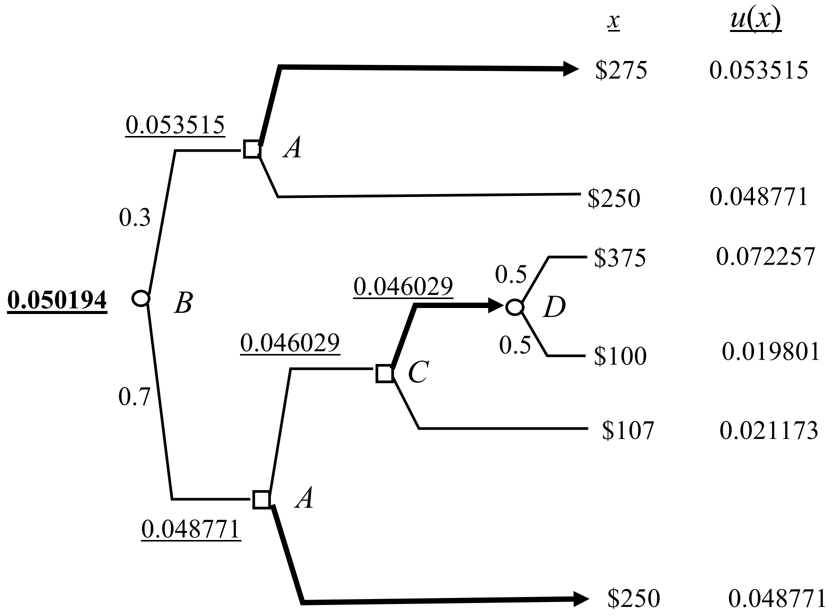


iv. Decision model with free clairvoyance on B before decision A.



- Expected Utility of free clairvoyance on B before making decision A = 0.050194
- Certainty Equivalent = u^{-1} (0.050194) = \$257.49
- Hence, the most Jeanne would pay for clairvoyance on B before A is \$257.49 – \$250.00 = \$7.49.

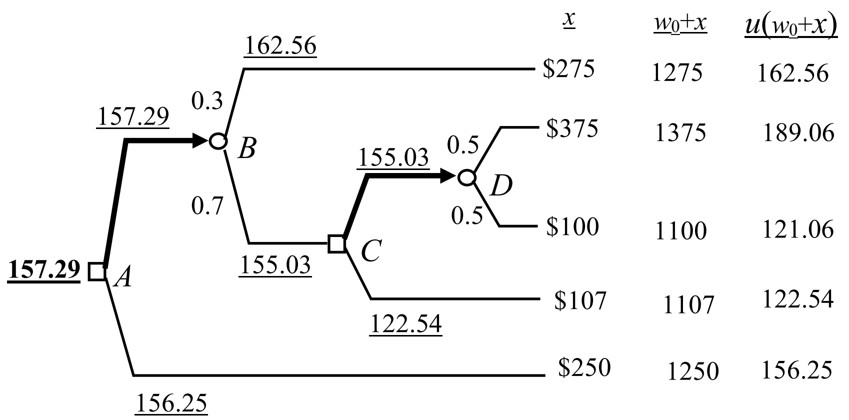
Simplified decision model without 0-1 probability nodes:



(*b*) Paulina utility function is $u(w) = \dfrac{w^2}{10000}$, and her current wealth is \$1,000.

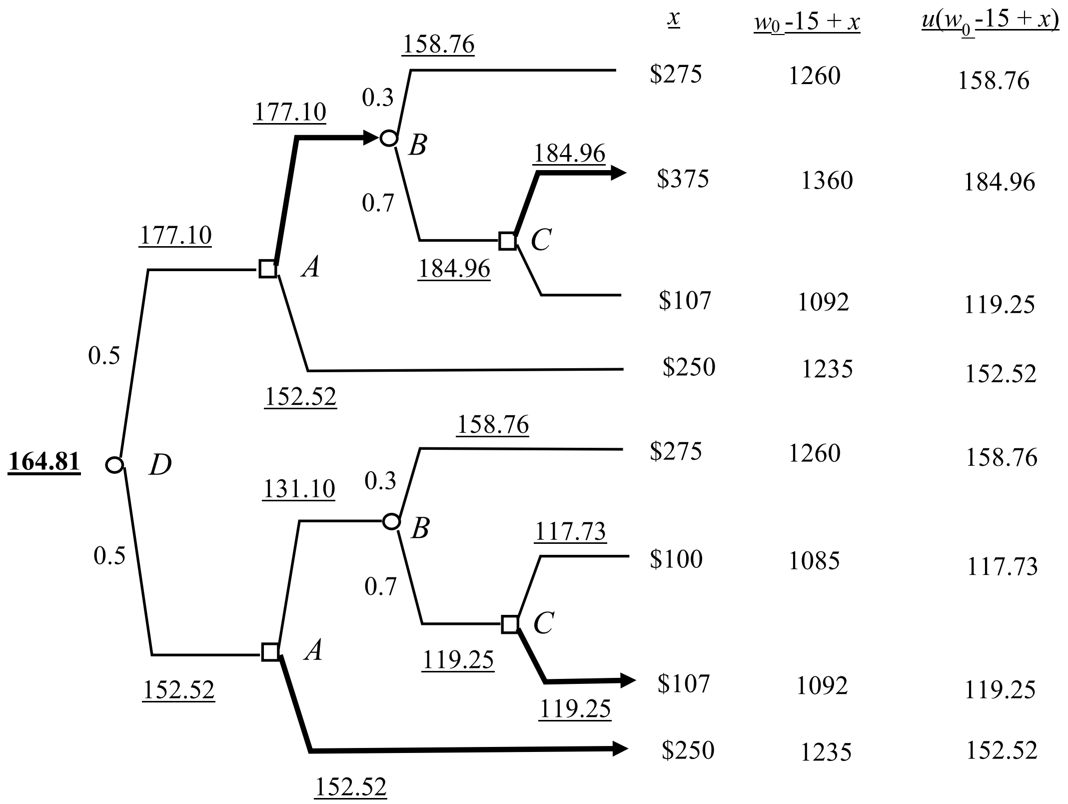
Note that Paulina does not follow the delta property as her utility function is neither exponential nor linear. Hence we need to consider Paulina's wealth in the analysis.

i. Base Decision Model:



- Maximum Expected Wealth Utility = 157.29.
- Wealth Certainty Equivalent = u^{-1} (157.29) = \$ 1,254.31.
- Certainty Equivalent for decision A is \$ 1,254.31 – \$1,000 = \$ 254.31.

ii. Decision model if Paulina pays \$15 for clairvoyance on *D* before making decision *A*:

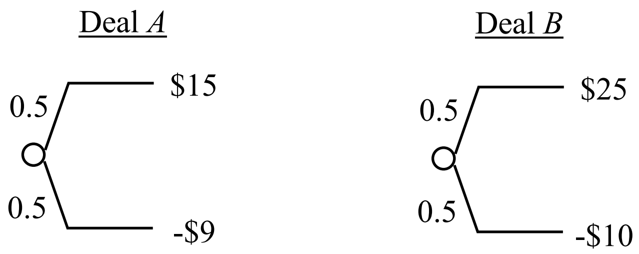


- Expected utility when Paulina pays \$15 for clairvoyance on *D* before making decision *A* = 164.81
- Expected utility without information = 157.29 < 164.81
- Hence, Paulina should pay \$15 for clairvoyance on D.

Note that Paulina does not satisfy the delta property. Therefore, we cannot use the "CE with free clairvoyance – CE no clairvoyance" method to compute her value of clairvoyance for any of the uncertain variables.

P6.8

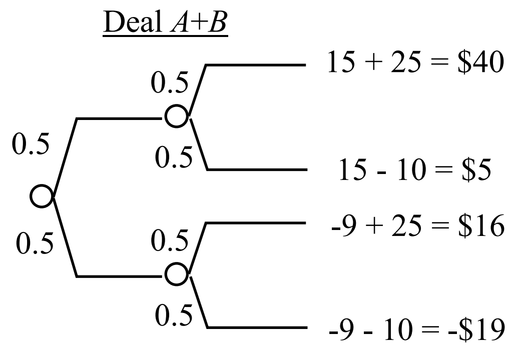
(*a*) Stan is risk neutral with initial wealth = \$20.



i. Stan's certainty equivalent for Deal *A* = (0.5)(15) + (0.5)(–9) = \$3
Since the certainty equivalent > \$0, Stan should accept Deal *A*.

ii. Stan's certainty equivalent for Deal *B* = (0.5)(25) + (0.5)(–10) = \$7.50.
Yes, Stan should accept Deal *B* also since its certainty equivalent > \$0.

iii. If Deals *A* and *B* are bundled, the situation appears as follows:

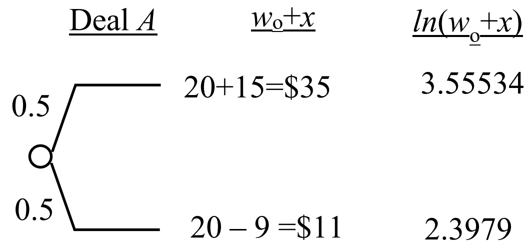


Stan's certainty equivalent for the bundle = (0.25) (40 + 5+ 16 – 19) = \$10.50

Note that this is exactly CE(*A*) + CE(*B*) = \$3.00 + \$7.50 = \$10.50 (See P6.3 for the general case with delta property).

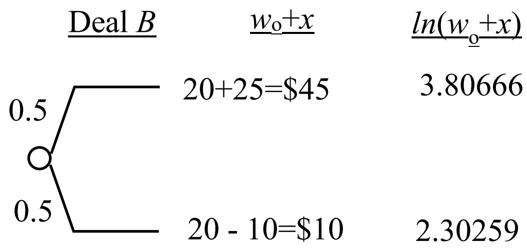
(*b*) Burke: $u(w)=ln(w)$, where w is total wealth. Initial wealth w_o = \$20.

i. Burke is offered Deal *A*:



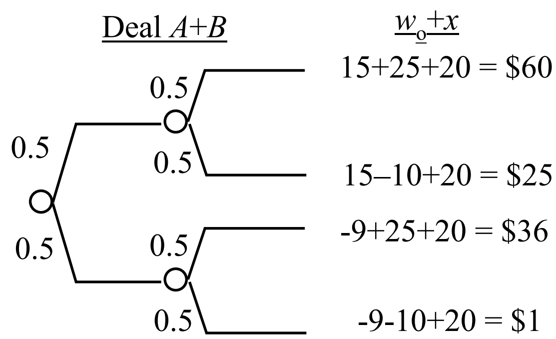
Expected utility for Deal *A* = (0.5) *ln*(35) + (0.5) *ln*(11) = 2.9766
Wealth CE for *A* = $u^{-1}(2.9766)$ = *exp*(2.9766) = \$19.62
CE for *A* = Wealth CE – Initial wealth = \$19.62 – \$20.00 = –\$0.38 < 0
Since the Certainty Equivalent of *A* < 0, Burke should **not** accept Deal *A*.

ii. Burke is offered Deal *B*:



Expected wealth utility for Deal *B* = (0.5) *ln*(45) + (0.5) *ln*(10) = 3.0546
Wealth CE for *B* = $u^{-1}(3.0546)$ = *exp*(3.0546) = \$21.21
CE for *A* = Wealth CE - Initial wealth = \$21.21 – \$20.00 = \$1.21 > 0
Since the Certainty Equivalent of *B* > 0, Burke should accept Deal *B*.

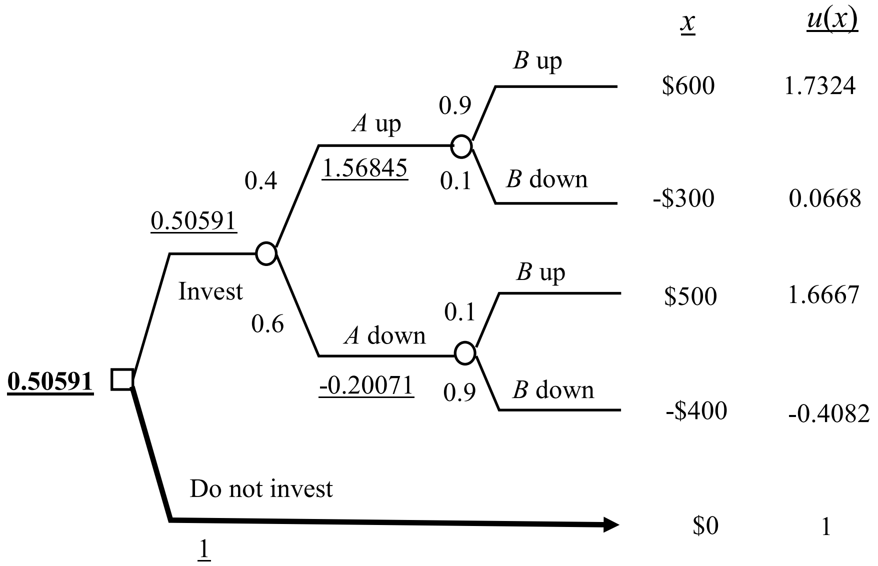
iii. If Deals *A* and *B* are bundled, the situation appears as follows:



Expected Wealth Utility for bundle *A*+*B* = (0.25)(*ln*(60) + *ln*(25) + *ln*(36) + *ln*(1)) = 2.724
Wealth CE for bundle = *exp* (2.724) = \$15.24
CE for bundle = \$15.24 – \$20.00 = –\$4.76 < 0
Therefore Burke should not accept the bundled deal.

(*c*) For Stan (risk neutral ⇒ delta property), CE(*A*) + CE(*B*) = CE(*A*+*B*).
For Burke (non delta property), CE(*A*) + CE(*B*) ≠ CE(*A*+*B*).

P6.9 Kay utility function is $u(x) = 2 - 9^{\frac{-x}{1000}} \Rightarrow$ Delta property.



(***a***) Expected utility for "Invest" = 0.50591

Expected utility for "Do not invest" = 1

(***b***) Expected dollar value for "Invest"

$= (0.4)(0.9 \times 600 + 0.1 \times -300) + (0.6)(0.1 \times 500 + 0.9 \times -400) = \$18.$

Expected dollar value for "Do not invest" = \$0.

(***c***) Since *EU*(Do not invest) > *EU*(Invest), Kay's best decision is "Do not invest"
Certainty Equivalent = \$0.

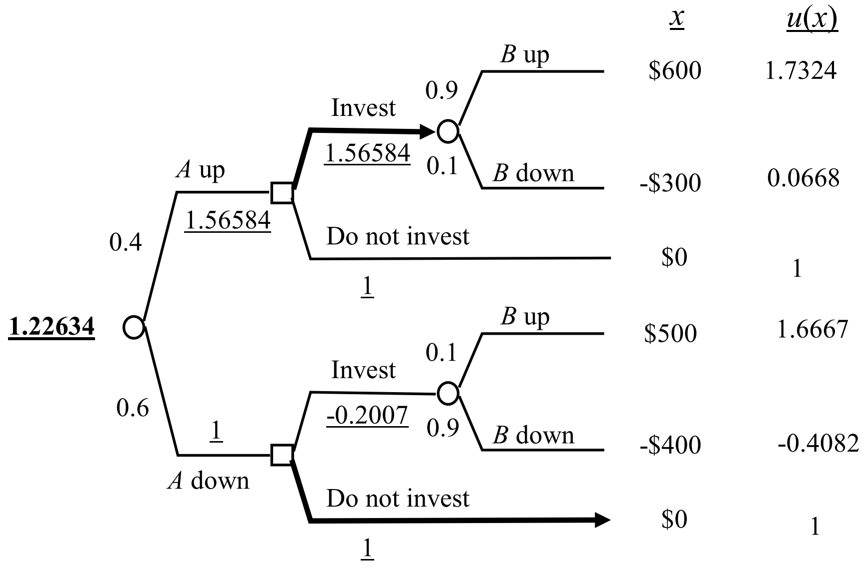
(***d***) We use the answers found in part (*a*) and not part (*b*) because Kay is not risk neutral (even though she satisfies the delta property).

We have to compare expected utilities and not expected dollar values. It would be okay to use expected dollar values if Kay was risk neutral.

If we had used expected dollar values instead, we would have ignored Kay's risk attitude and may get a wrong answer since expected dollar values are not the same as the certainty equivalents.

(*e*) Since Kay satisfies the delta property, we can easily find her value of clairvoyance on *A* using the difference of CEs method.

Decision model with free clairvoyance on Stock *A*:

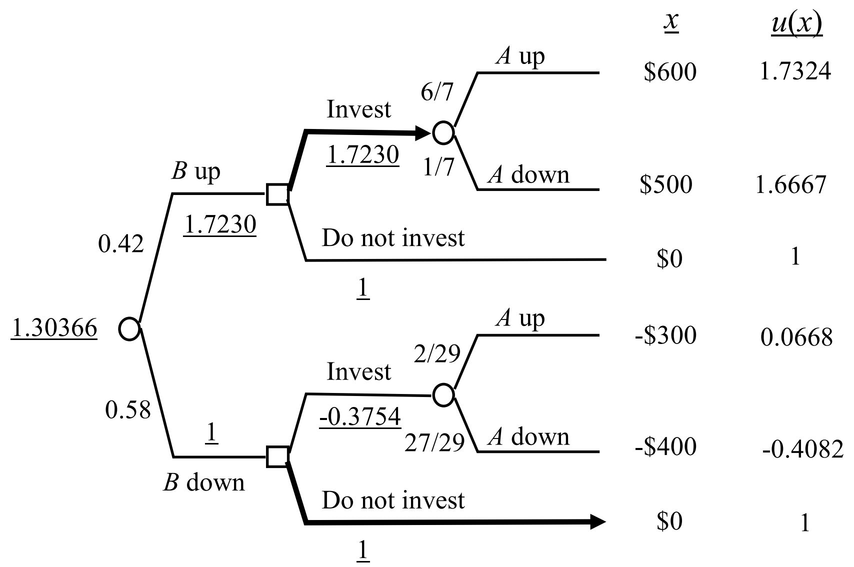


Expected Utility with free clairvoyance = 1.22634.

To find the certainty equivalent: $2-9^{-\frac{x}{1000}}=1.22634 \Rightarrow x=\$116.8$

Certainty equivalent with free clairvoyance on *A* = \$116.80
Hence value of clairvoyance on *A* = \$116.80 –\$ 0 = \$116.80 > \$10.
Therefore Kay should pay \$10 for clairvoyance on *A*.

(*f*) Decision model with free clairvoyance on Stock *B*:



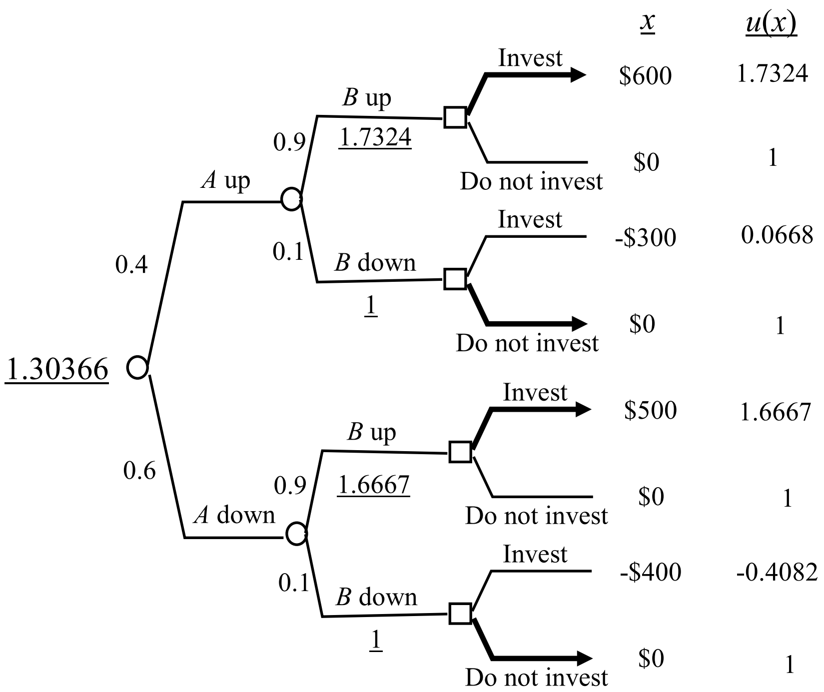
Expected utility with free clairvoyance on *B* = 1.30366

$2-9^{-\frac{x}{1000}}=1.30366 \Rightarrow x=\$164.71$

Certainty equivalent with free clairvoyance on *B* = \$164.71

Hence value of clairvoyance on *B* = \$164.71 –\$ 0 = \$164.71 > \$10.
Therefore Kay should pay \$10 for clairvoyance on *B*.

(g) Decision model with free joint clairvoyance on Stock A & Stock B:



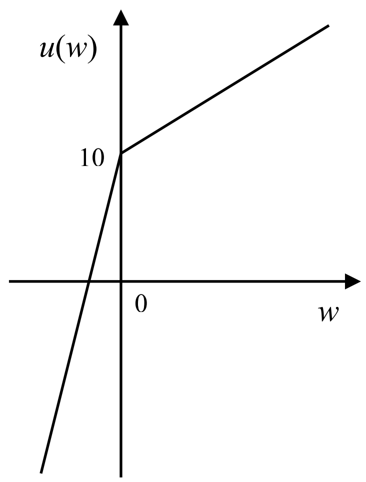
Expected utility with free clairvoyance on both A and B = 1.30366

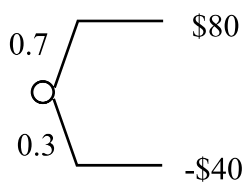
Certainty Equivalent with free clairvoyance on both A and B = u^{-1} (1.30366) = $164.71

Kay's value of clairvoyance on A and B together = $164.71 –$ 0 = $164.71.

P6.10 Daniel: $u(w) = \begin{cases} w+10, & w \geq 0 \\ 6w+10, & w < 0 \end{cases}$, where w is total assets in dollars. w_o = \$20.

Mad Dog: $u(x) = 2^{\frac{x}{50}}$; w_o = \$75. Note that Mad Dog has the delta property.





(***a***) Daniel's expected wealth utility = (0.7) u(80 + 20) + (0.3) u(–40 + 20)
= (0.7) u(100) + (0.3) u(–20)
= (0.7) (110) + (0.3)(–110) = 44

Wealth Certainty Equivalent = 44 – 10 = \$34.00

Daniel's Certainty Equivalent for deal = \$34.00 – \$20.00 = \$14.00

(***b***) Daniel's personal indifferent selling price is \$14**.**00

We need to compute Mad Dog's personal indifferent buying price. Since Mad Dog has an exponential utility function, he satisfies the delta property.

Furthermore, personal indifferent buying price = personal indifferent selling price = certainty equivalent.

CE for Mad Dog = u^{-1} (0.7 u(80) + 0.3 u(–40))
= u^{-1} (0.7 (2^(80/50)) + 0.3 (2^(–40/50)))
= u^{-1} (2.2943)
= \$59.90
= Personal indifferent buying price for Mad Dog.

Therefore, we can make a maximum of \$59.90 – \$14.00 = \$45.90 by buying the deal from Daniel and selling it to Mad Dog.